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Entropy Change in Gravitational Collapse

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Gravitational collapse is an isoentropic process for an isolated perfect fluid and the entropy decreases for an open collapsing system. We also give a distinguishing parameter for the entropy change for an imperfect fluid. By use of the distinguishing parameter, we calculate the entropy changes of self-gravitational collapsing systems and conclude that the total entropy of a collapsing system decreases or is unchanged before the system's horizon appears.

1. INTRODUCTION

Why is the entropy of a black hole so high? This question has been in the air for 20 years now. Some authors (Sorkin *et al.*, 1981; Gui and Liu, 1982; Zurek and Page, 1984; Zhang, 1986) have used the entropy of a self-gravitational collapsing system to approach the entropy of the black hole into which the system collapses. They found entropies to increase.

But we find that in this work there are common problems:

1. The collapsing systems were regarded as isolated systems.
2. Metrics for the collapsing systems were assumed static.
3. The high entropy of a black hole stems from the collapse process before the black hole forms.

In fact, a star must emit various particles in its evolution. Especially after going through the red giant phase it can throw out numerous objects. A supernova explosion is just like this. The matter which forms a black hole is not all the matter of the collapsing system, but a part of it, and even a small part. Therefore we would like to regard the collapsing system as an open system rather than an isolated system.

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Second, collapse is a dynamic nonequilibrium process. The metric inside the system obviously ought to include time. Adopting a static metric for the collapsing system is evidently incorrect.

In this paper we will calculate the entropy changes of collapsing systems using a dynamic metric.

2. THE CASE OF PERFECT FLUID

First we consider an open system with heat flow such as a system with escaping neutrinos. We take the stress-energy tensor of the system to be

$$T^{ab} = pg^{ab} + (\rho + p)u^a u^b + q^a u^b + q^b u^a$$

where q^a is the heat flow four-vector, in the comoving coordinate $q^a = (0, q, 0, 0)$, $q_a u^a = 0$. If C is the cooling rate for a unit amount of matter, or the rate of decrease of internal energy due to the neutrino emission, then

$$-nC = -u^a[pg^b_a + (\rho + p)u_a u^b]_{;b} = (\rho u^a)_{;a} + pu^a_{;a} \tag{1}$$

Using the relation $\rho = n(1 + \varepsilon)$, we rewrite (1) as

$$D_t \varepsilon = -C - pD_t(1/n) \tag{2}$$

Here ε is the specific internal energy, and D_t is differentiation with respect to the proper time, $D_t = u^a\, \partial/\partial x^a$. Considering the thermodynamic law $d\varepsilon = T\, ds - pd(1/n)$, we have

$$TD_t s = -C \tag{3}$$

The metric with spherical symmetry is chosen as

$$ds^2 = -dt^2 + B^2(r, t)(dr^2 + r^2 d\Omega^2) \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, and $(-g)^{1/2} = B^3 r^2 \sin\theta$, $u^a = (1, 0, 0, 0)$. In the comoving coordinate, the equation of continuity becomes

$$D_t(nB^3 r^2) = 0 \tag{5}$$

Therefore, at the moment t for a spherical layer of the open collapsing system the rate of change of entropy is

$$\begin{aligned} D_t \mathscr{S}_t &= D_t(4\pi r^2 B^3 ns\, dr) \\ &= 4\pi[nB^3 r^2 D_t s + sD_t(nB^3 r^2)]\, dr \\ &= -4CnB^3 r^2\, dr/T < 0 \end{aligned} \tag{6}$$

Here $\mathscr{S}_t$ is the entropy of the fluid layer with radius r and thickness dr. From (6) we conclude that in self-gravitational collapse with radial heat flux the total entropy of the system decreases gradually.

3. THE CASE OF IMPERFECT FLUID

We assume that the stress-energy tensor of the imperfect fluid is

$$T^{ab} = pg^{ab} + (p + \rho)u^a u^b + \Delta T^{ab} \tag{7}$$

where the modified term ΔT^{ab} is (Weinberg, 1972)

$$\Delta T^{ab} = -\eta H^{ac} H^{bd} W_{cd} - K(H^{ac}u^b + H^{bc}u^a)Q_c - \xi H^{ab} u^c_{;c} \tag{8}$$

W_{ab} is the shear tensor, $W_{ab} = u_{a;b} + u_{b;a} - \frac{2}{3} g_{ab} u^c_{;c}$.
Q_a is the thermal flux four-vector, $Q_a = T_{;a} + Tu_{a;b}u^b$.
H_{ab} are the projection tensors, $H_{ab} = g_{ab} + u_a u_b$.
K, η, and ξ are the thermal conductivity coefficient, shear viscosity coefficient, and volume viscosity coefficient, respectively.

According to the relations

$$g_{ab}u^a u^b = -1, \qquad u_a u^a_{;b} = 0$$

$$K_B T\, d\sigma = p\, d(1/n) + d(\rho/n)$$

$$(T^{ab})_{;b} = 0, \qquad N^a_{;a} = (nu^a)_{;a} = 0$$

together with the thermodynamic law and the conservation law, we have

$$(nK_B \sigma u^b)_{;b} = u_a (\Delta T^{ab})_{;b}/T \tag{9}$$

Here σ is the specific entropy, K_B is the Boltzmann constant, and n is the particle number density.

In the comoving frame of reference, the four-vector of the entropy flux is

$$S^a = nK_B \sigma u^a - T^{-1} u_b\, \Delta T^{ba} \tag{10}$$

We take the comoving metric to be

$$ds^2 = -(dx^0)^2 + A(dx^1)^2 + B(dx^2)^2 + C(dx^3)^2 \tag{11}$$

where A, B, C are functions of x^a.

From equations (9) and (10), we get

$$(nK_B \sigma u^b)_{;b} = (-g)^{-1/2} (nK_B \sigma u^0 (-g)^{1/2})_{,0} \tag{12}$$

The total entropy of the collapsing system can be written as

$$S_T = \int S(-g)^{1/2}\, dx^1\, dx^2\, dx^3 = \int (nK_B \sigma)(-g)^{1/2}\, dx^1\, dx^2\, dx^3 \tag{13}$$

In the comoving frame,

$$D_t S_T = u^a S_{T,a} = S_{T,0}$$

Using equation (12), we have

$$D_t S_T = \int (-g)^{1/2} (nK_B \sigma u^b)_{;b} \, dx^1 \, dx^2 \, dx^3 \tag{14}$$

where $(-g)^{1/2} > 0$, $dx^1 \, dx^2 \, dx^3 > 0$, the sign of $D_t S_T$ is only defined from that of Δ, and

$$\begin{aligned} \Delta &= (nK_B \sigma u^b)_{;b} \\ &= T^{-1} u_0 (\Delta T^{0b})_{;b} \end{aligned} \tag{15}$$

The total entropy increases when $\Delta > 0$, decreases when $\Delta < 0$, and is conserved when $\Delta = 0$. So we call Δ the distinguishing parameter of entropy change of the collapsing system.

4. DISCUSSION AND COMMENT

1. For $K = \eta = \xi = 0$, we have

$$T^{ab} = 0 \qquad \text{and} \qquad T^{ab} = (p + \rho) u^a u^b + p g^{ab}$$

From (15), $\Delta = 0$.

2. For $\eta \neq 0$, $K = \xi = 0$, the stress-energy tensor becomes

$$T^{ab} = (p + \rho) u^a u^b + p g^{ab} - \eta H^{ac} H^{bd} W_{cd}$$

Calculation easily gives

$$\Delta = -(2/3)\eta(\dot{A}/A + \dot{B}/B + \dot{C}/C)^2 / TD < 0$$

3. For $\xi \neq 0$, $K = \eta = 0$, from

$$T^{ab} = (p + \rho) u^a u^b + p g^{ab} - \xi H^{ab} u^c_{;c}$$

we have

$$\Delta = \xi(\dot{A}/A + \dot{B}/B + \dot{C}/C)^2 / TD > 0$$

But in astrophysics $\xi \ll \eta$, i.e., ξ can be regarded as zero when $\eta = 0$. Therefore this case is not important.

4. For $\eta \neq 0$, $\xi \neq 0$, $K = 0$,

$$T^{ab} = (p + \rho) u^a u^b + p g^{ab} - \eta H^{ac} H^{bd} W_{cd} - \xi H^{ab} u^c_{;c}$$

and

$$\Delta = (\xi - 2\eta/3)(\dot{A}/A + \dot{B}/B + \dot{C}/C)^2 / TD$$

If $\xi = 2\eta/3$, then $\Delta = 0$. If $\xi < 2\eta/3$, then $\Delta < 0$.

5. For $K \neq 0$, $\xi = \eta = 0$,

$$T^{ab} = (p+\rho)u^a u^b + pg^{ab} - K(H^{ac}u^b + H^{bc}u^a)Q_c$$

Adopting Misner's (1965) model, we have

$$\Delta = -nC/T < 0$$

where

$$C = -u_a(\Delta T^{ab})_{;b}/n$$

and for the model of de Oliveira and Santos (1985) we have

$$\Delta = -q(2A'/A + 3B'/B + 2/r)/T < 0$$

From the above discussions, we conclude that the total entropy of the collapsing system decreases or is unchanged before the system's horizon appears. Therefore the high entropy of a black hole is dependent on the horizon. We will discuss the mechanism for this high entropy in another paper.

REFERENCES

De Oliveira, A. K. G., and Santos, N. O. (1985). *Monthly Notices of the Royal Astronomical Society*, **216**, 403, 1001.

Misner, C. W. (1965). *Physical Review*, **137**, B1360.

Sorkin, R. D., Wald, R. W., and Zhang, Z. J. (1981). *General Relativity and Gravitation*, **13**, 1127.

Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.

Gui Yuanxin, and Liu Liao (1982). *Acta Astrophysica Sinica*, **2**, 97.

Zhang, Z. J. (1986). *Modern Relativity and Black Hole Physics*, Publishing House of Huazhong Normal University, p. 363.

Zurek, W. H., and Page, D. N. (1984). *Physical Review*, **29**, 628.